Estimates for parameters and characteristics of the confining SU(3)-gluonic field in π^0-meson from one- and two-photon decays

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Abstract

On the basis of the confinement mechanism earlier proposed by author the electric formfactor of π^0-meson is nonperturbatively calculated. The latter is then applied to describing electromagnetic decays $\pi^0 \to 2\gamma$ and $\pi^0 \to e^+e^-\gamma$ which entails estimates for parameters of the confining SU(3)-gluonic field in π^0-meson. The corresponding estimates of the gluon concentrations, electric and magnetic colour field strengths are also adduced for the mentioned field.

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1 Introduction and preliminary remarks

In Refs. [1–3] for the Dirac-Yang-Mills system derived from QCD-Lagrangian there was found and explored an unique family of compatible nonperturbative solutions which could pretend to decsribing confinement of two quarks. Applications of the family to description of both the heavy quarkonia spectra [4] and a number of properties of pions and kaons [5] showed that the confinement mechanism is qualitatively the same for both light mesons and heavy quarkonia.

Two main physical reasons for linear confinement in the mechanism under discussion are the following ones. The first one is that gluon exchange between quarks is realized with the propagator different from the photon one and existence and form of such a propagator is *direct* consequence of the unique confining nonperturbative solutions of the Yang-Mills equations [2,3]. The second reason is that, owing to the structure of mentioned propagator,

quarks mainly emit and interchange the soft gluons so the gluon condensate (a classical gluon field) between quarks basically consists of soft gluons (for more details see Refs. [2,3]) but, because of that any gluon also emits gluons (still softer), the corresponding gluon concentrations rapidly become huge and form the linear confining magnetic colour field of enormous strengths which leads to confinement of quarks. This is by virtue of the fact that just magnetic part of the mentioned propagator is responsible for larger portion of gluon concentrations at large distances since the magnetic part has stronger infrared singularities than the electric one. Under the circumstances physically nonlinearity of the Yang-Mills equations effectively vanishes so the latter possess the unique nonperturbative confining solutions of the Abelian-like form (with the values in Cartan subalgebra of SU(3)-Lie algebra) [2,3] that describe the gluon condensate under consideration. Moreover, since the overwhelming majority of gluons is soft they cannot leave hadron (meson) until some gluon obtains additional energy (due to an external reason) to rush out. So we deal with confinement of gluons as well.

The approach under discussion equips us with the explicit wave functions for every two quarks (meson). The wave functions are parametrized by a set of real constants a_j, b_j, B_j describing the mentioned *nonperturbative* confining gluon field (the gluon condensate) and they are *nonperturbative* modulo square integrable solutions of the Dirac equation in the above confining SU(3)-field and also depend on μ_0, the reduced mass of the current masses of quarks forming meson. It is clear that under the given approach just constants a_j, b_j, B_j, μ_0 determine all properties of any meson and they should be extracted from experimental data.

Such a program has been to a certain extent advanced in Refs. [4,5]. The aim of the present paper is to continue obtaining estimates for a_j, b_j, B_j for concrete mesons starting from experimental data on spectroscopy of one or another meson. We here consider π^0-meson and its electromagnetic decays $\pi^0 \to 2\gamma$ and $\pi^0 \to e^+e^-\gamma$ whose widths practically add up to the full width of π^0-meson [6].

Of course, when conducting our considerations we shall rely on the standard quark model (SQM) based on SU(3)-flavor symmetry (see, e. g., Ref. [6]) so in accordance with SQM $\pi^0 = (\overline{u}u - \overline{d}d)/\sqrt{2}$ is equiprobable superposition of two quarkonia, consequently, we shall have two sets of parameters a_j, b_j, B_j.

Further we shall deal with the metric of the flat Minkowski spacetime M that we write down (using the ordinary set of local spherical coordinates r, ϑ, φ for the spatial part) in the form

$$ds^2 = g_{\mu\nu}dx^\mu \otimes dx^\nu \equiv dt^2 - dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2). \qquad (1)$$

Besides, we have $|\delta| = |\det(g_{\mu\nu})| = (r^2\sin\vartheta)^2$ and $0 \le r < \infty$, $0 \le \vartheta < \pi$,

$0 \leq \varphi < 2\pi$.

Throughout the paper we employ the Heaviside-Lorentz system of units with $\hbar = c = 1$, unless explicitly stated otherwise, so the gauge coupling constant g and the strong coupling constant α_s are connected by relation $g^2/(4\pi) = \alpha_s$. Further we shall denote $L_2(F)$ the set of the modulo square integrable complex functions on any manifold F furnished with an integration measure, then $L_2^n(F)$ will be the n-fold direct product of $L_2(F)$ endowed with the obvious scalar product while † and ∗ stand, respectively, for Hermitian and complex conjugation. Our choice of Dirac γ-matrices conforms to the so-called standard representation and is the same as in Ref. [5]. At last $\otimes$ means tensorial product of matrices and I_3 is the unit 3×3 matrix.

When calculating we apply the relations $1 \text{ GeV}^{-1} \approx 0.1973269679 \text{ fm}$, $1 \text{ s}^{-1} \approx 0.658211915 \times 10^{-24} \text{ GeV}$, $1 \text{ V/m} \approx 0.2309956375 \times 10^{-23} \text{ GeV}^2$, $1 \text{ T} \approx 0.6925075988 \times 10^{-15} \text{ GeV}^2$.

Finally, for the necessary estimates we shall employ the T_{00}-component (volumetric energy density) of the energy-momentum tensor for a SU(3)-Yang-Mills field which should be written in the chosen system of units in the form

$$T_{\mu\nu} = -F_{\mu\alpha}^a \, F_{\nu\beta}^a \, g^{\alpha\beta} + \frac{1}{4} F_{\beta\gamma}^a \, F_{\alpha\delta}^a g^{\alpha\beta} g^{\gamma\delta} g_{\mu\nu} \, . \tag{2}$$

2   Specification of main relations

As was mentioned above, our considerations shall be based on the unique family of compatible nonperturbative solutions for the Dirac-Yang-Mills system (derived from QCD-Lagrangian) studied in details in Refs. [1–3]. Referring for more details to those references, let us briefly decribe and specify only the relations necessary to us in the present Letter.

One part of the mentioned family is presented by the unique nonperturbative confining solution of the Yang-Mills equations for $A = A_\mu dx^\mu = A_\mu^a \lambda_a dx^\mu$ (λ_a are the known Gell-Mann matrices, $\mu = t, r, \vartheta, \varphi$, $a = 1, ..., 8$) and looks as follows

$$\mathcal{A}_{1t} \equiv A_t^3 + \frac{1}{\sqrt{3}} A_t^8 = -\frac{a_1}{r} + A_1 \, , \mathcal{A}_{2t} \equiv -A_t^3 + \frac{1}{\sqrt{3}} A_t^8 = -\frac{a_2}{r} + A_2 \, ,$$

$$\mathcal{A}_{3t} \equiv -\frac{2}{\sqrt{3}} A_t^8 = \frac{a_1 + a_2}{r} - (A_1 + A_2) \, ,$$

$$\mathcal{A}_{1\varphi} \equiv A_\varphi^3 + \frac{1}{\sqrt{3}} A_\varphi^8 = b_1 r + B_1 \, , \mathcal{A}_{2\varphi} \equiv -A_\varphi^3 + \frac{1}{\sqrt{3}} A_\varphi^8 = b_2 r + B_2 \, ,$$

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$$\mathcal{A}_{3\varphi} \equiv -\frac{2}{\sqrt{3}} A_\varphi^8 = -(b_1 + b_2)r - (B_1 + B_2) \tag{3}$$

with the real constants a_j, A_j, b_j, B_j parametrizing the family. As has been repeatedly explained in Refs. [2–5], parameters $A_{1,2}$ of solution (3) are inessential for physics in question and we can consider $A_1 = A_2 = 0$. Obviously we have $\sum_{j=1}^{3} \mathcal{A}_{jt} = \sum_{j=1}^{3} \mathcal{A}_{j\varphi} = 0$ which reflects the fact that for any matrix $\mathcal{T}$ from SU(3)-Lie algebra we have $\text{Tr}\,\mathcal{T} = 0$.

Another part of the family is given by the unique nonperturbative modulo square integrable solutions of the Dirac equation in the confining SU(3)-field of (3) $\Psi = (\Psi_1, \Psi_2, \Psi_3)$ with the four-dimensional Dirac spinors Ψ_j representing the jth colour component of the meson, so Ψ may describe relative motion (relativistic bound states) of two quarks in mesons and the mentioned Dirac equation looks as follows

$$i\partial_t \Psi \equiv i \begin{pmatrix} \partial_t \Psi_1 \\ \partial_t \Psi_2 \\ \partial_t \Psi_3 \end{pmatrix} = H\Psi \equiv \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = \begin{pmatrix} H_1\Psi_1 \\ H_2\Psi_2 \\ H_3\Psi_3 \end{pmatrix}, \tag{4}$$

where Hamiltonian H_j is

$$H_j = \gamma^0 \left[\mu_0 - i\gamma^1 \partial_r - i\gamma^2 \frac{1}{r}\left(\partial_\vartheta + \frac{1}{2}\gamma^1\gamma^2\right) - i\gamma^3 \frac{1}{r\sin\vartheta}\left(\partial_\varphi + \frac{1}{2}\sin\vartheta\gamma^1\gamma^3 + \frac{1}{2}\cos\vartheta\gamma^2\gamma^3\right) \right]$$

$$- g\gamma^0 \left(\gamma^0 \mathcal{A}_{jt} + \gamma^3 \frac{1}{r\sin\vartheta}\mathcal{A}_{j\varphi}\right) \tag{5}$$

with the gauge coupling constant g while μ_0 is a mass parameter and one can consider it to be the reduced mass which is equal, *e. g.*, for quarkonia, to half the current mass of quarks forming a quarkonium.

Then the unique nonperturbative modulo square integrable solutions of (4) are (with Pauli matrix σ_1)

$$\Psi_j = e^{-i\omega_j t}\psi_j \equiv e^{-i\omega_j t} r^{-1} \begin{pmatrix} F_{j1}(r)\Phi_j(\vartheta, \varphi) \\ F_{j2}(r)\sigma_1\Phi_j(\vartheta, \varphi) \end{pmatrix}, j = 1, 2, 3 \tag{6}$$

with the 2D eigenspinor $\Phi_j = \begin{pmatrix} \Phi_{j1} \\ \Phi_{j2} \end{pmatrix}$ of the euclidean Dirac operator $\mathcal{D}_0$ on the unit sphere $\mathbb{S}^2$, while the coordinate r stands for the distance between quarks. The explicit form of Φ_j is not needed here and can be found in Refs. [3,7]. For the purpose of the present Letter we shall adduce the necessary spinors below. Spinors Φ_j form an orthonormal basis in $L_2^2(\mathbb{S}^2)$. We can call the quantity ω_j relative energy of jth colour component of meson (while ψ_j is wave function of a stationary state for jth colour component) but we can see that if we want to interpret (4) as equation for eigenvalues of the relative motion energy, i. e., to rewrite it in the form $H\psi = \omega\psi$ with $\psi = (\psi_1, \psi_2, \psi_3)$ then we should put $\omega = \omega_j$ for any j so that $H_j\psi_j = \omega_j\psi_j = \omega\psi_j$. Under this

situation, if a meson is composed of quarks $q_{1,2}$ with different flavours then the energy spectrum of the meson will be given by $\epsilon = m_{q_1} + m_{q_2} + \omega$ with the current quark masses m_{q_k} (rest energies) of the corresponding quarks. On the other hand for determination of ω_j the following quadratic equation can be obtained [1–3]

$$[g^2 a_j^2 + (n_j + \alpha_j)^2]\omega_j^2 - 2(\lambda_j - gB_j)g^2 a_j b_j\,\omega_j + [(\lambda_j - gB_j)^2 - (n_j + \alpha_j)^2]g^2 b_j^2 - \mu_0^2(n_j + \alpha_j)^2 = 0, \tag{7}$$

that yields

$$\omega_j = \omega_j(n_j, l_j, \lambda_j) =$$
$$\frac{\Lambda_j g^2 a_j b_j \pm (n_j + \alpha_j)\sqrt{(n_j^2 + 2n_j\alpha_j + \Lambda_j^2)\mu_0^2 + g^2 b_j^2(n_j^2 + 2n_j\alpha_j)}}{n_j^2 + 2n_j\alpha_j + \Lambda_j^2}, j = 1, 2, 3\,, \tag{8}$$

where $a_3 = -(a_1 + a_2)$, $b_3 = -(b_1 + b_2)$, $B_3 = -(B_1 + B_2)$, $\Lambda_j = \lambda_j - gB_j$, $\alpha_j = \sqrt{\Lambda_j^2 - g^2 a_j^2}$, $n_j = 0, 1, 2, ...$, while $\lambda_j = \pm(l_j + 1)$ are the eigenvalues of euclidean Dirac operator $\mathcal{D}_0$ on unit sphere with $l_j = 0, 1, 2, ...$ It should be noted that in previous papers [1–5] we used the ansatz (6) with the factor $e^{i\omega_j t}$ instead of $e^{-i\omega_j t}$ but then the Dirac equation (4) would look as $-i\partial_t\Psi = H\Psi$ and in equation (7) the second summand would have plus sign while the first summand in numerator of (8) would have minus sign. We here return to the conventional form of writing Dirac equation and this slightly modifies the equations (7)–(8). In line with the above we should have $\omega = \omega_1 = \omega_2 = \omega_3$ in energy spectrum $\epsilon = m_{q_1} + m_{q_2} + \omega$ for any meson and this at once imposes two conditions on parameters a_j, b_j, B_j when choosing some experimental value for ϵ at the given current quark masses m_{q_1}, m_{q_2}.

Within the given Letter we need only the radial parts of (6) at $n_j = 0$ (the ground state) that are

$$F_{j1} = C_j P_j r^{\alpha_j} e^{-\beta_j r}\left(1 - \frac{gb_j}{\beta_j}\right), P_j = gb_j + \beta_j,$$

$$F_{j2} = iC_j Q_j r^{\alpha_j} e^{-\beta_j r}\left(1 + \frac{gb_j}{\beta_j}\right), Q_j = \mu_0 - \omega_j \tag{9}$$

with $\beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2 b_j^2}$ while C_j is determined from the normalization condition $\int_0^\infty(|F_{j1}|^2 + |F_{j2}|^2)dr = \frac{1}{3}$. Consequently, we shall gain that $\Psi_j \in L_2^4(\mathbb{R}^3)$ at any $t \in \mathbb{R}$ and, as a result, the solutions of (6) may describe relativistic bound states (mesons) with the energy (mass) spectrum ϵ.

It is useful to specify the nonrelativistic limit (when $c \to \infty$) for spectrum (8). For that one should replace $g \to g/\sqrt{\hbar c}$, $a_j \to a_j/\sqrt{\hbar c}$, $b_j \to b_j\sqrt{\hbar c}$, $B_j \to B_j/\sqrt{\hbar c}$ and, expanding (8) in $z = 1/c$, we shall get

$$\omega_j(n_j, l_j, \lambda_j) =$$

$$\pm\mu_0 c^2 \left[1 \mp \frac{g^2 a_j^2}{2\hbar^2(n_j + |\lambda_j|)^2}z^2\right] + \left[\frac{\lambda_j g^2 a_j b_j}{\hbar(n_j + |\lambda_j|)^2} \mp \mu_0 \frac{g^3 B_j a_j^2 f(n_j, \lambda_j)}{\hbar^3(n_j + |\lambda_j|)^7}\right] z + O(z^2),$$

(10)

where $f(n_j, \lambda_j) = 4\lambda_j n_j(n_j^2 + \lambda_j^2) + \frac{|\lambda_j|}{\lambda_j}\left(n_j^4 + 6n_j^2\lambda_j^2 + \lambda_j^4\right).$

As is seen from (10), at $c \to \infty$ the contribution of linear magnetic colour field (parameters b_j, B_j) to spectrum really vanishes and spectrum in essence becomes purely Coulomb one (modulo the rest energy). Also it is clear that when $n_j \to \infty$, $\omega_j \to \pm\sqrt{\mu_0^2 + g^2 b_j^2}$.

We may seemingly use (8) with various combinations of signes ($\pm$) before second summand in numerators of (8) but, due to (10), it is reasonable to take all signs equal to plus which is our choice within the Letter. Besides, as is not complicated to see, radial parts in nonrelativistic limit have the behaviour of form $F_{j1}, F_{j2} \sim r^{l_j+1}$, which allows one to call quantum number l_j angular momentum for jth colour component though angular momentum is not conserved in the field (3) [1,3]. So for meson under consideration we should put all $l_j = 0$.

Finally it should be noted that spectrum (8) is degenerated owing to degeneracy of eigenvalues for the euclidean Dirac operator $\mathcal{D}_0$ on the unit sphere $\mathbb{S}^2$. Namely, each eigenvlalue of $\mathcal{D}_0$ $\lambda = \pm(l + 1), l = 0, 1, 2...$, has multiplicity $2(l + 1)$ so we has $2(l + 1)$ eigenspinors orthogonal to each other. Ad referendum we need eigenspinors corresponding to $\lambda = \pm 1$ ($l = 0$) so here is their explicit form

$$\lambda = -1 : \Phi = \frac{C}{2}\begin{pmatrix} e^{i\frac{\vartheta}{2}} \\ e^{-i\frac{\vartheta}{2}} \end{pmatrix} e^{i\varphi/2}, \text{ or } \Phi = \frac{C}{2}\begin{pmatrix} e^{i\frac{\vartheta}{2}} \\ -e^{-i\frac{\vartheta}{2}} \end{pmatrix} e^{-i\varphi/2},$$

$$\lambda = 1 : \Phi = \frac{C}{2}\begin{pmatrix} e^{-i\frac{\vartheta}{2}} \\ e^{i\frac{\vartheta}{2}} \end{pmatrix} e^{i\varphi/2}, \text{ or } \Phi = \frac{C}{2}\begin{pmatrix} -e^{-i\frac{\vartheta}{2}} \\ e^{i\frac{\vartheta}{2}} \end{pmatrix} e^{-i\varphi/2} \quad (11)$$

with the coefficient $C = 1/\sqrt{2\pi}$ (for more details see Refs. [3,7]).

3 Electric and magnetic formfactors, anomalous magnetic moment

As has been mentioned in Section 1, we shall analyse electromagnetic decays of π^0-meson so let us consider what quantities characterizing electromagnetic properties of a meson we could construct within our approach.

Within the present Letter we shall use relations (8) at $n_j = 0 = l_j$ so energy (mass) of meson under consideration is given by $\mu = 2m_q + \omega$ with $\omega =$

$\omega_j(0, 0, \lambda_j)$ for any $j = 1, 2, 3$ whereas

$$\omega = \frac{g^2 a_1 b_1}{\Lambda_1} + \frac{\alpha_1 \mu_0}{|\Lambda_1|} = \frac{g^2 a_2 b_2}{\Lambda_2} + \frac{\alpha_2 \mu_0}{|\Lambda_2|} = \frac{g^2 a_3 b_3}{\Lambda_3} + \frac{\alpha_3 \mu_0}{|\Lambda_3|} = \mu - 2m_q \quad (12)$$

and, as a consequence, the corresponding meson wave functions of (6) are represented by (9), (11).

3.1 Choice of quark masses and gauge coupling constant

It is evident for employing the above relations we have to assign some values to quark masses and gauge coupling constant g. In accordance with Ref. [6], at present the current quark masses necessary to us are restricted to intervals 1.5 MeV $\le m_u \le$ 3 MeV, 3.0 MeV $\le m_d \le$ 7 MeV, so we take $m_u = (1.5 + 3)/2$ MeV $= 2.25$ MeV, $m_d = (3 + 7)/2$ MeV $= 5$ MeV. Under the circumstances, the reduced mass μ_0 of (5) will respectively take values $m_u/2, m_d/2$. As to gauge coupling constant $g = \sqrt{4\pi\alpha_s}$, it should be noted that recently some attempts have been made to generalize the standard formula for $\alpha_s = \alpha_s(Q^2) = 12\pi/[(33 - 2n_f)\ln(Q^2/\Lambda^2)]$ (n_f is number of quark flavours) holding true at the momentum transfer $\sqrt{Q^2} \to \infty$ to the whole interval $0 \le \sqrt{Q^2} \le \infty$. We shall employ one such a generalization [8] used in Refs. [9]. It looks as follows ($x = \sqrt{Q^2}$ in GeV)

$$\alpha(x) = \frac{12\pi}{(33 - 2n_f)} \frac{f_1(x)}{\ln \frac{x^2 + f_2(x)}{\Lambda^2}} \quad (13)$$

with

$$f_1(x) = 1 + \left(\left(\frac{(1 + x)(33 - 2n_f)}{12}\ln\frac{m^2}{\Lambda^2} - 1\right)^{-1} + 0.6x^{1.3}\right)^{-1}, f_2(x) = m^2(1 + 2.8x^2)^{-2},$$

wherefrom one can conclude that $\alpha_s \to \pi = 3.1415...$ when $x \to 0$, i. e., $g \to 2\pi = 6.2831....$ We used (13) at $m = 1$ GeV, $\Lambda = 0.234$ GeV, $n_f = 3$, $x = m_{\pi^0} = 134.976$ MeV to obtain $g = 6.101482007$ necessary for our further computations at the mass scale of π^0-meson.

3.2 Electric formfactor

For each meson with the wave function $\Psi = (\Psi_j)$ of (6) we can define electromagnetic current $J^\mu = \overline{\Psi}(\gamma^\mu \otimes I_3)\Psi = (\Psi^\dagger\Psi, \Psi^\dagger(\alpha \otimes I_3)\Psi) = (\rho, \mathbf{J})$, $\alpha = \gamma^0\gamma$.

Electric formfactor $f(K)$ is the Fourier transform of ρ

$$f(K) = \int \Psi^\dagger \Psi e^{-i\mathbf{Kr}} d^3x = \sum_{j=1}^{3} \int \Psi_j^\dagger \Psi_j e^{-i\mathbf{Kr}} d^3x = \sum_{j=1}^{3} f_j(K) =$$

$$\sum_{j=1}^{3} \int (|F_{j1}|^2 + |F_{j2}|^2) \Phi_j^\dagger \Phi_j \frac{e^{-i\mathbf{Kr}}}{r^2} d^3x, \ d^3x = r^2 \sin\vartheta dr d\vartheta d\varphi \qquad (14)$$

with the momentum transfer K. At $n_j = 0 = l_j$, as is easily seen, for any spinor of (11) we have $\Phi_j^\dagger \Phi_j = 1/(4\pi)$, so the integrand in (14) does not depend on φ and we can consider vector $\mathbf{K}$ to be directed along z-axis. Then $\mathbf{Kr} = Kr\cos\vartheta$ and with the help of (9) and relations (see Ref. [10]): $\int_0^\infty r^{\alpha-1} e^{-pr} dr = \Gamma(\alpha) p^{-\alpha}$, Re $\alpha, p > 0$, $\int_0^\infty r^{\alpha-1} e^{-pr} \begin{pmatrix} \sin(Kr) \\ \cos(Kr) \end{pmatrix} dr = \Gamma(\alpha)(K^2 + p^2)^{-\alpha/2} \begin{pmatrix} \sin(\alpha \arctan(K/p)) \\ \cos(\alpha \arctan(K/p)) \end{pmatrix}$, Re $\alpha > -1$, Re $p > |\mathrm{Im}\, K|$, $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $\int_0^\pi e^{-iKr\cos\vartheta} \sin\vartheta d\vartheta = 2\sin(Kr)/(Kr)$, we shall obtain

$$f(K) = \sum_{j=1}^{3} f_j(K) = \sum_{j=1}^{3} \frac{(2\beta_j)^{2\alpha_j+1}}{6\alpha_j} \cdot \frac{\sin[2\alpha_j \arctan(K/(2\beta_j))]}{K(K^2 + 4\beta_j^2)^{\alpha_j}}$$

$$= \sum_{j=1}^{3} \left(\frac{1}{3} - \frac{2\alpha_j^2 + 3\alpha_j + 1}{6\beta_j^2} \cdot \frac{K^2}{6} \right) + O(K^4), \qquad (15)$$

wherefrom it is clear that $f(K)$ is a function of K^2, as should be, and we can determine the root-mean-square radius of meson in the form

$$< r > = \sqrt{\sum_{j=1}^{3} \frac{2\alpha_j^2 + 3\alpha_j + 1}{6\beta_j^2}}. \qquad (16)$$

It is clear, we can directly calculate $< r >$ in accordance with the standard quantum mechanics rules as $< r > = \sqrt{\int r^2 \Psi^\dagger \Psi d^3x} = \sqrt{\sum_{j=1}^{3} \int r^2 \Psi_j^\dagger \Psi_j d^3x}$ and the result will be the same as in (16). So we should not call $< r >$ of (16) the *charge* radius of meson – it is just the radius of meson determined by the wave functions of (6) (at $n_j = 0 = l_j$) with respect to strong interaction. Now we should notice the expression (15) to depend on 3-vector $\mathbf{K}$. To rewrite it in the form holding true for any 4-vector Q, let us remind that according to general considerations (see, e.g., Ref. [11]) the relation (15) corresponds to the so-called Breit frame where $Q^2 = -K^2$ [when fixing metric by (1)] so it is not complicated to rewrite (15) for arbitrary Q in the form

$$f(Q^2) = \sum_{j=1}^{3} f_j(Q^2) = \sum_{j=1}^{3} \frac{(2\beta_j)^{2\alpha_j+1}}{6\alpha_j} \cdot \frac{\sin[2\alpha_j \arctan(\sqrt{|Q^2|}/(2\beta_j))]}{\sqrt{|Q^2|}(4\beta_j^2 - Q^2)^{\alpha_j}} \qquad (17)$$

which passes on to (15) in the Breit frame.

We can define the volumetric magnetic moment density by $\mathbf{m} = q(\mathbf{r} \times \mathbf{J})/2 = q[(yJ_z - zJ_y)\mathbf{i} + (zJ_x - xJ_z)\mathbf{j} + (xJ_y - yJ_x)\mathbf{k}]/2$ with the meson charge q and $\mathbf{J} = \Psi^\dagger(\alpha \otimes I_3)\Psi$. Using (6) we have in the explicit form

$$J_x = \sum_{j=1}^{3}(F_{j1}^* F_{j2} + F_{j2}^* F_{j1})\frac{\Phi_j^\dagger \Phi_j}{r^2}, \; J_y = \sum_{j=1}^{3}(F_{j1}^* F_{j2} - F_{j2}^* F_{j1})\frac{\Phi_j^\dagger \sigma_2 \sigma_1 \Phi_j}{r^2},$$

$$J_z = \sum_{j=1}^{3}(F_{j1}^* F_{j2} - F_{j2}^* F_{j1})\frac{\Phi_j^\dagger \sigma_3 \sigma_1 \Phi_j}{r^2} \tag{18}$$

with Pauli matrices $\sigma_{1,2,3}$. Magnetic moment of meson is $\mathbf{M} = \int_V \mathbf{m} d^3 x$, where V is volume of meson (the ball of radius $< r >$). Then at $n_j = l_j = 0$, as is seen from (9), (11), $F_{j1}^* = F_{j1}, F_{j2}^* = -F_{j2}, \Phi_j^\dagger \sigma_2 \sigma_1 \Phi_j = 0$ for any spinor of (11) which entails $J_x = J_y = 0$, i.e., $m_z = 0$ while $\int_V m_{x,y} d^3 x = 0$ because of turning the integral over φ to zero, which is easy to check. As a result, magnetic moments of mesons with the wave functions of (6) (at $l_j = 0$) are equal to zero, as should be according to experimental data [6].

We can, however, define magnetic formfactor $F(K)$ of meson if noticing that

$$\int_V \sqrt{m_x^2 + m_y^2 + m_z^2}\,d^3 x \neq \sqrt{(\int_V m_x d^3 x)^2 + (\int_V m_y d^3 x)^2 + (\int_V m_z d^3 x)^2} = M = 0.$$

Under the circumstances we should define $F(K)$ as the inverse Fourier transform of $\sqrt{m_x^2 + m_y^2 + m_z^2}$

$$\sqrt{m_x^2 + m_y^2 + m_z^2} = \frac{q}{2\mu(2\pi)^3}\int F(K)e^{i\mathbf{Kr}}d^3 K$$

with meson mass μ so that

$$F(K) = \frac{2\mu}{q}\int\sqrt{m_x^2 + m_y^2 + m_z^2}e^{-i\mathbf{Kr}}d^3 x = \mu\int r|J_z \sin\vartheta|e^{-i\mathbf{Kr}}d^3 x =$$

$$\frac{\mu}{2\pi}\int\sum_{j=1}^{3}|F_{j1}||F_{j2}|\frac{\sin^2\vartheta}{r}e^{-i\mathbf{Kr}}d^3 x \tag{19}$$

for any spinor of (11) so the integrand in (19) does not again depend on φ. Then $\mathbf{Kr} = Kr\cos\vartheta$ and using the relation $\int_0^\pi e^{-iKr\cos\vartheta}\sin^3\vartheta d\vartheta = 4[\sin(Kr)/(Kr)^3 - \cos(Kr)/(Kr)^2]$, we shall obtain

$$F(K) = 4\mu\int_0^\infty r\sum_{j=1}^{3}C_j^2 P_j Q_j r^{2\alpha_j}e^{-2\beta_j r}\left(1 - \frac{g^2 b_j^2}{\beta_j^2}\right)\left[\frac{\sin(Kr)}{(Kr)^3} - \frac{\cos(Kr)}{(Kr)^2}\right]dr =$$

$$\frac{2\mu}{3}\sum_{j=1}^{3}\frac{(2\beta_j)^{2\alpha_j+1}}{\alpha_j}\cdot\frac{P_jQ_j(1-\frac{g^2b_j^2}{\beta_j^2})}{(1-\frac{gb_j}{\beta_j})^2P_j^2+(1+\frac{gb_j}{\beta_j})^2Q_j^2}\cdot\frac{1}{K^2(K^2+4\beta_j^2)^{\alpha_j}}\cdot$$

$$\left\{\frac{\sin\left[(2\alpha_j-1)\arctan\left(K/(2\beta_j)\right)\right]}{(2\alpha_j-1)K(K^2+4\beta_j^2)^{-1/2}}-\cos\left[2\alpha_j\arctan\left(K/(2\beta_j)\right)\right]\right\}=$$

$$\frac{4\mu}{3}\sum_{j=1}^{3}\frac{\beta_j}{\alpha_j}\cdot\frac{P_jQ_j(1-\frac{g^2b_j^2}{\beta_j^2})}{(1-\frac{gb_j}{\beta_j})^2P_j^2+(1+\frac{gb_j}{\beta_j})^2Q_j^2}\left[\frac{\alpha_j(2\alpha_j+1)}{6\beta_j^2}-\frac{\alpha_j(4\alpha_j^3+12\alpha_j^2+11\alpha_j+3)}{120\beta_j^4}K^2+O(K^4)\right],$$

$$(20)$$

where the fact was used that by virtue of the normalization condition for wave functions we have $C_j^2[P_j^2(1-gb_j/\beta_j)^2+Q_j^2(1+gb_j/\beta_j)^2]=(2\beta_j)^{2\alpha_j+1}/[3\Gamma(2\alpha_j+1)]$. It is clear from (20) that $F(K)$ is a function of K^2 and we can rewrite (20) for arbitrary 4-vector Q by the same manner as was done for electric formfactor $f(K)$ in (17).

Now we can define *the anomalous magnetic moment density* for meson by the relation

$$m_a=\lim_{K\to0}\frac{q}{2\mu(2\pi)^3}\int F(K)e^{i\mathbf{K}\mathbf{r}}d^3K=\frac{q}{2\mu}F(0)\delta(\mathbf{r}),$$

so *anomalous magnetic moment* is

$$M_a=\int_V m_ad^3x\approx\int_{\mathbb{R}^3}m_ad^3x=\frac{q}{2\mu}F(0)\qquad(21)$$

with

$$F(0)=\frac{4\mu}{3}\sum_{j=1}^{3}\frac{2\alpha_j+1}{6\beta_j}\cdot\frac{P_jQ_j(1-\frac{g^2b_j^2}{\beta_j^2})}{(1-\frac{gb_j}{\beta_j})^2P_j^2+(1+\frac{gb_j}{\beta_j})^2Q_j^2},$$

as follows from (20). It is clear that for neutral mesons $M_a\equiv0$ due to $q=0$. It should be noted that a possibility of existence of anomalous magnetic moments for mesons is permanently discussed in literature (see, e.g., Refs. [12] and references therein) though there is no experimental evidence in this direction [6].

4   Estimates for parameters of SU(3)-gluonic field in π^0-meson

4.1   Basic equations

The question now is how to apply the obtained formfactors to electromagnetic decays $\pi^0\to2\gamma$ and $\pi^0\to e^+e^-\gamma$ to estimate parameters of SU(3)-gluonic field in π^0-meson. The corresponding diagrams for decays under consideration are presented in Fig. 1. Let us at first consider two-photon decay (Fig. 1a).

Actually kinematic analysis based on Lorentz- and gauge invariances gives rise to the following expression for width Γ_1 of the given decay (see, e.g., Refs. [13])

$$\Gamma_1 = \frac{1}{4}\pi\alpha_{em}^2 g_{\pi\gamma\gamma}^2 \mu^3 \tag{22}$$

with electromagnetic coupling constant α_{em}=1/137.0359895 and π^0-meson mass $\mu = 134.976$ MeV while the information about strong interaction of quarks in π^0-meson is encoded in a decay constant $g_{\pi\gamma\gamma}$. Making replacement $g_{\pi\gamma\gamma} = \sqrt{2}/(4\pi^2 f_P)$ we can reduce (22) to the more conventional form (see, e. g., Refs. [14])

$$\Gamma_1 = \frac{\alpha_{em}^2 \mu^3}{32\pi^3 f_P^2} \, , \tag{23}$$

so the present-day experimental value $\Gamma_1 \approx 7.741$ eV conforms to $f_P \approx 130$ MeV [6]. Further parametrization in the form $f_P = \mu\Phi$ entails $\Phi \approx 0.9673759912$. We can now notice that the only invariant which Φ might depend on is $Q^2 = \mu^2$, i. e. we should find such a function $f(Q^2)$ for that $f(Q^2 = \mu^2) \approx 0.9673759912$. It is obvious from physical point of view that f should be connected with electromagnetic properties of π^0-meson. As we have seen above in Section 3, there are at least two suitable functions for this aim – electric and magnetic formfactors. But, as was mentioned, there exist no experimental consequences related to magnetic formfactor at present whereas electric one to some extent determines, e. g., an effective size of meson in the form $< r >$ of (16). It is reasonable, therefore, to take just electric formfactor of (17) for the sought function. Then, denoting the quantities $\mu/(2\beta_j) = x_j$, we obtain the following equation for parameters of the confining SU(3)-gluonic field in π^0-meson

$$f(Q^2 = \mu^2) = \sum_{j=1}^{3} f_j(Q^2 = \mu^2) = \sum_{j=1}^{3} \frac{1}{6\alpha_j x_j} \cdot \frac{\sin\left(2\alpha_j \arctan x_j\right)}{(1 - x_j^2)^{\alpha_j}} \approx 0.9673759912. \tag{24}$$

One more equation can be gained from consideration of one-photon decay (Fig. 1b) with width $\Gamma_2 \approx 0.0939$ eV. The latter in crucial way depends on the value of electric formfactor of π^0-meson at $Q^2 = (2m_e)^2$, where $m_e = 0.510998918$ MeV is electron mass (for more details see Ref. [6] and book by Pilkuhn in [13]). The present-day value for $f(Q^2 = 4m_e^2)$ is between 0.9999936936 and 1.000001835 [6]. So, when designating $m_e/\beta_j = y_j$, we obtain the mentioned equation in the form

$$f(Q^2 = 4m_e^2) = \sum_{j=1}^{3} f_j(Q^2 = 4m_e^2) = \sum_{j=1}^{3} \frac{1}{6\alpha_j y_j} \cdot \frac{\sin\left(2\alpha_j \arctan y_j\right)}{(1 - y_j^2)^{\alpha_j}} \approx 1.00. \tag{25}$$

At last, equations (12) and (16) should also be added to (24), (25) and the system obtained in such a way should be solved compatibly if taking $< r >$=0.672 fm [6].

Fig. 1. Two- and one-photon electromagnetic decays of π^0-meson

4.2 Numerical results

The results of numerical compatible solving of equations (12), (16), (24) and (25) are adduced in Tables 1–3.

Table 1

Gauge coupling constant, mass parameter μ_0 and parameters of the confining SU(3)-gluonic field for π^0-meson

Particle	g	μ_0 (MeV)	a_1	a_2	b_1 (GeV)	b_2 (GeV)	B_1	B_2
π^0—$\overline{u}u$	6.10148	1.125	-0.0434737	-0.00680835	0.0848234	0.0433136	0.01	-0.150
π^0—$\overline{d}d$	6.10148	2.50	-0.0606679	0.0251427	0.0956303	0.0648174	0.1250	-0.2450

Table 2

Theoretical and experimental π^0-meson mass and radius

Particle	Theoret. μ (MeV)	Experim. μ (MeV)	Theoret. $< r >$ (fm)	Experim. $< r >$ (fm)
π^0—$\overline{u}u$	$\mu = 2m_u + \omega_j(0,0,-1) = 134.976$	134.976	0.674927	0.672
π^0—$\overline{d}d$	$\mu = 2m_d + \omega_j(0,0,-1) = 134.976$	134.976	0.675676	0.672

Table 3

Theoretical and experimental π^0-meson electric formfactor values

Particle	Theoret. $f(Q^2 = \mu^2)$	Experim. $f(Q^2 = \mu^2)$	Theoret. $f(Q^2 = 4m_e^2)$	Experim. $f(Q^2 = 4m_e^2)$
π^0—$\overline{u}u$	0.991172	0.967376	0.999999464	0.9999936936–1.000001835
π^0—$\overline{d}d$	0.996480	0.967376	0.999999762	0.9999936936–1.000001835

5 Estimates of gluon concentrations, electric and magnetic colour field strengths

Now let us remind that, according to Refs. [3,5], one can confront the field (3) with T_{00}-component (volumetric energy density of the SU(3)-gluonic field) of the energy-momentum tensor (2) so that

$$T_{00} \equiv T_{tt} = \frac{E^2 + H^2}{2} = \frac{1}{2}\left(\frac{a_1^2 + a_1 a_2 + a_2^2}{r^4} + \frac{b_1^2 + b_1 b_2 + b_2^2}{r^2 \sin^2 \vartheta}\right) \equiv \frac{\mathcal{A}}{r^4} + \frac{\mathcal{B}}{r^2 \sin^2 \vartheta}$$

(26)

with electric E and magnetic H colour field strengths and with real $\mathcal{A} > 0$, $\mathcal{B} > 0$.

To estimate the gluon concentrations we can employ (26) and, taking the quantity $\omega = \Gamma$, the full decay width of a meson, for the characteristic frequency of gluons we obtain the sought characteristic concentration n in the

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form

$$n = \frac{T_{00}}{\Gamma} \tag{27}$$

so we can rewrite (26) in the form $T_{00} = T_{00}^{\text{coul}} + T_{00}^{\text{lin}}$ conforming to the contributions from the Coulomb and linear parts of the solution (3). This entails the corresponding split of n from (27) as $n = n_{\text{coul}} + n_{\text{lin}}$.

The parameters of Table 1 were employed when computing and for simplicity we put $\sin \vartheta = 1$ in (26). Also there was used the following present-day full decay width of π^0-meson: $\Gamma = 1/\tau$ with the life time $\tau = 8.4 \times 10^{-17}$ s, whereas the Bohr radius $a_0 = 0.529177249 \cdot 10^5$ fm [6].

Table 4 contains the numerical results for n_{coul}, n_{lin}, n, E, H for the meson under discussion.

Table 4
Gluon concentrations, electric and magnetic colour field strengths in π^0-meson

π^0—$\bar{u}u$:	$r_0 = <r> = 0.674927$ fm				
r	n_{coul}	n_{lin}	n	E	H
(fm)	(m^{-3})	(m^{-3})	(m^{-3})	(V/m)	(T)
$0.1r_0$	0.893407×10^{57}	0.232356×10^{56}	0.916643×10^{57}	0.174836×10^{24}	0.476625×10^{15}
r_0	0.893407×10^{53}	0.232356×10^{54}	0.321696×10^{54}	0.174836×10^{22}	0.476625×10^{14}
1.0	0.185386×10^{53}	0.105844×10^{54}	0.124383×10^{54}	0.796426×10^{21}	0.321687×10^{14}
$10r_0$	0.893407×10^{49}	0.232356×10^{52}	0.233249×10^{52}	0.174836×10^{20}	0.476625×10^{13}
a_0	0.236413×10^{34}	0.377976×10^{44}	0.377976×10^{44}	0.284409×10^{12}	0.607900×10^{9}
π^0—$\bar{d}d$:	$r_0 = <r> = 0.675676$ fm				
r	n_{coul}	n_{lin}	n	E	H
(fm)	(m^{-3})	(m^{-3})	(m^{-3})	(V/m)	(T)
$0.1r_0$	0.111063×10^{58}	0.355535×10^{56}	0.114618×10^{58}	0.194936×10^{24}	0.589577×10^{15}
r_0	0.111063×10^{54}	0.355535×10^{54}	0.466597×10^{54}	0.194936×10^{22}	0.589577×10^{14}
1.0	0.231485×10^{53}	0.162315×10^{54}	0.185464×10^{54}	0.889956×10^{21}	0.398363×10^{14}
$10r_0$	0.111063×10^{50}	0.355535×10^{52}	0.356645×10^{52}	0.194936×10^{20}	0.589577×10^{13}
a_0	0.295201×10^{34}	0.579638×10^{44}	0.579638×10^{44}	0.317809×10^{12}	0.752797×10^{9}

6 Discussion and concluding remarks

6.1 Discussion

As is seen from Table 4, at the characteristic scales of π^0-meson the gluon concentrations are huge and the corresponding fields (electric and magnetic colour ones) can be considered to be the classical ones with enormous strenghts. The part n_{coul} of gluon concentration n connected with the Coulomb electric colour field is decreasing faster than n_{lin}, the part of n related to the linear magnetic colour field, and at large distances n_{lin} becomes dominant. It should be emphasized that in fact the gluon concentrations are much greater than the estimates given in Table 4 because the latter are the estimates for maximal possible gluon frequencies, i.e. for maximal possible gluon impulses (under the concrete situation of π^0-meson). The given picture is in concordance with the one obtained in Refs. [4,5]. As a result, the confinement mechanism developed in Refs. [1–3] is also confirmed by the considerations of the present Letter.

It should be noted, however, that our results are of a preliminary character which is readily apparent, for example, from that the current quark masses (as well as the gauge coupling constant g) used in computation are known only within the certain limits and we can expect similar limits for the magnitudes discussed in the Letter so it is neccesary further specification of the parameters for the confining SU(3)-gluonic field in π^0-meson which can be obtained, for instance, by calculating weak formfactors in processes $K^+ \rightarrow \pi^0 + l^+ + \nu_l$ ($l = e, \mu$) with the help of wave functions discussed above. We hope to continue analysing the given problems elsewhere.

At last it should be noted that the relation (23) is often connected with the so-called chiral (axial) anomaly and PCAC model (see for more details, e.g., Refs. [14]). Under the latter approach, however, the value of f_P in (23) is all the same taken from experimental data while we express it through more fundamental parameters connected with the unique exact solution (3) of Yang-Mills equations describing confinement. But, e. g., in the case of decay $\eta \rightarrow 2\gamma$ the PCAC model is unlikely to be applicable and our approach works since it is based on the unique family of compatible nonperturbative solutions for the Dirac-Yang-Mills system derived from QCD-Lagrangian and, as a result, the approach is itself nonperturbative and may be employed for any meson.

6.2 Concluding remarks

Finally we should note the following. Our approach actually repeats the one of quantum electrodynamics (QED) at large distances: nonperturbative effects

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of quark interaction are described by the modulo square integrable solutions of Dirac equation in the classical confining SU(3)-gluonic field (condensate of huge number of gluons) and for this aim we use the unique compatible non-perturbative solutions of the Dirac-Yang-Mills system derived directly from QCD-Lagrangian which is in perfect analogy with QED where one uses the unique modulo square integrable solutions of Dirac (Schrödinger) equation in the Coulomb field (condensate of huge number of photons), i. e., one employs the unique compatible nonperturbative solutions of the Maxwell-Dirac (Schrödinger) system directly derived from QED-Lagrangian to describe, for example, the hydrogen atom spectrum. But the same analogy with QED prompts to us that further study should perhaps include radiative corrections. This should, however, be conditioned by experiment. At the moment, as far as is known to us, little is known about such effects. At any rate, as seems to us, our approach treats nonperturbative meson physics in a sound way. Another matter that QCD-Lagrangian probably allows one to develop some other approaches (not based on compatible solutions of the Dirac-Yang-Mills system) to confinement but our one appears to be rather natural since it is practically identical to the standard approach of quantum mechanics and QED to description of bound states in external (electromagnetic) fields. In other words, this approach should have been one of the very first approaches as soon as the QCD-Lagrangian was written. Historically, however, this way was rejected due to incomprehensible reasons without any detailed study.

Acknowledgements

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